

# Theory of cooling by flow through narrow pores

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We consider the possibility of adding a stage to a dilution refrigerator to provide additional cooling by “filtering out” hot atoms. Three methods are considered: 1) Effusion, where holes having diameters larger than a mean-free path allow atoms to pass through easily; 2) Particle waveguide-like motion using very narrow channels that greatly restrict the quantum states of the atoms in a channel. 3) Wall-limited diffusion through channels, in which the wall scattering is disordered so that local density equilibrium is established in a channel. We assume that channel dimension are smaller than the mean-free path for atom-atom interactions. The particle waveguide and the wall-limited diffusion methods using channels on order of the de Broglie wavelength give cooling. Recent advances in nano-filters give this method some hope of being practical.

We investigate here the possibility cooling a gas by “filtering out” hot atoms, perhaps as an adjunct to a dilution refrigerator with solutions of  $^3\text{He}$  in liquid  $^4\text{He}$ . [1] Our method involves passing a degenerate Fermi gas through narrow constrictions formed by pores in a partition. Under certain conditions we find that this particle “leakage” allows a lowering of the temperature of the remaining gas. Our study adds to many recent experimental and theoretical analyses of quantum size effects in the behavior of particles in nanopores (as in Refs. [2, 3, 4]).

An initial idea of how to remove the hot atoms is suggested by a technique used in electron heterojunction physics, [5, 6, 7, 8] where the electron gas is passed through a narrow constriction formed by a gate potential. Because the constriction is narrow the bands of states allowed in this “particle waveguide” are widely separated, which means that not all energies are allowed through. Adjusting the states in the channel can allow selective passage particles in states at the Fermi energy, so that one removes only hot atoms. In such an approach the constriction must be of order of the de Broglie wavelength of atoms at the top of the Fermi surface, which is roughly the separation between fermions. Because of the limited states (that is, bands) we characterize channels of this size as “*narrow*.” We will see that this approach can be made to work under appropriate conditions on the nano-pores.

An alternative possibility involves channels or pores with diameters much larger than the de Broglie wavelength but still less than the mean-free path of the fermions in the gas, which can be large due to its  $1/T^2$  behavior. We call these “*wide*” pores.

We will consider different kinds of flow through the holes of the two sizes mentioned: *effusion*, *waveguide flow*, and *wall-limited diffusion*. In effusion, the holes are by definition wide and the states in the channel remain three-dimensional with no banding. Moreover the walls are sufficiently smooth that the particles undergo no

back scattering nor do they come into equilibrium with the channel walls. Effectively all that the pores do is to allow particles already directed in the positive  $z$ -direction to pass through the membrane. We might hope that this would allow cooling because the intensity of fast atoms passing through a hole is larger than that of the slow atoms and these, on average, carry more energy. Indeed it is a standard textbook exercise [10] to show that this works for a Boltzmann gas. But does it work for a degenerate Fermi gas?

In waveguide flow the channels are narrow enough to have well-defined bands. Again the particles are assumed to undergo no back scattering nor do they come into thermal equilibrium with the channel walls. Nevertheless because of the bands only particles with certain energies are allowed in the channel. If, for example, the Fermi energy of the gas in the container is coincident with the bottom of the lowest band one might expect that only high energy particles would get through the holes and the remaining gas would be cooled.

A third situation is wall-limited diffusion or Knudsen flow, which could occur in wide or narrow pores. There is scattering at the walls including back scattering. The rate of diffusion in the channel depends on the diameter of the pore, the particle velocity, and the density gradient that maintains the flow. In this and the above methods we will assume the density difference is maintained by pumping away the particles that pass through the membrane. Both effusion and wall-limited diffusion were considered by one of the authors some time ago for enhancing polarization. [9]

If the Fermi gas in the cooling cell is, say, a 1% solution of  $^3\text{He}$  in liquid  $^4\text{He}$  at millikelvin temperature, then the Fermi temperature is 124 mK, the de Broglie wavelength about 3 nm, and the mean-free path is about  $1\text{ }\mu\text{m}$  at  $T = 15\text{ mK}$ .

Suppose the number of atoms in a box (B1) is  $N$  with density  $n$ . The atoms pass through a membrane and en-

ter a second box (B2) maintained (by pumping) at a much smaller particle number  $N_2$  and density  $n_2$ . The membrane contains a great number  $M$  of holes each of diameter  $d$  and of total hole area  $A$ . (See Fig. 1.) We will compute particle current  $J_N$  and energy current  $J_E$  passing through the holes. The rates of particle and energy change,  $dN/dt = AJ_N$  and  $dE/dt = AJ_E$  in the container B1 combine in

$$\frac{dE}{dt} = C_V \frac{dT}{dt} + \left( \frac{\partial E}{\partial N} \right)_{T,V} \frac{dN}{dt} \quad (1)$$

to give the cooling power  $C_V dT/dt$  where  $C_V$  is the heat capacity at constant volume in B1.

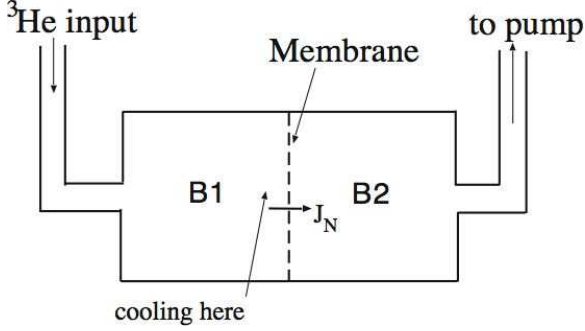


FIG. 1: Schematic diagram of the apparatus.  $^3\text{He}$  in solution with liquid  $^4\text{He}$  enters the cooling chamber B1, passes selectively through the membrane into B2. A gas at lowered temperature remains in B1. The  $^3\text{He}$  density in B2 is kept low by pumping. The gas is recycled, after being recooled by a dilution fridge, back into B1.

In the case of effusion the particle current is

$$J_N(1 \rightarrow 2) = \frac{2}{h^3} \int_{-\infty}^{+\infty} dp_x \int_{-\infty}^{+\infty} dp_y \int_0^{+\infty} dp_z \frac{p_z}{m} n_p \quad (2)$$

where  $n_p$  is the momentum distribution function and the factor of 2 accounts for spin degeneracy. There is a similar expression for  $J_E$ . For a classical gas passing through wide holes we find a cooling power given by  $-nk_B AT (k_B T/2\pi m)^{1/2}$ . The minus sign implies cooling in agreement with the usual textbook treatment.[10] In the degenerate limit effusive cooling fails.

For particle waveguide motion (in narrow holes) we need to take into account the banding of the states due to the limited transverse motion; the energies are given by  $\epsilon_{nz} = \epsilon_n + \epsilon_z$  with  $\epsilon_z = \hbar^2 k_z^2/2m$ . We will assume a square cross section with width  $d$  so that  $\epsilon_n = \epsilon_0 \eta_n$  where  $\epsilon_0 = \pi^2 \hbar^2/(md^2)$  and  $\eta_n = 1, 2.5, \dots, (n_x^2 + n_y^2)/2, \dots$  (Using a circular cross section gives no significant differences in our final results.) In this case Eq. (2) is replaced by

$$J_N(1 \rightarrow 2) = \frac{2}{d^2 h} \sum_n \int_0^{+\infty} dp_z \frac{p_z}{m} \frac{1}{(e^{\beta(\epsilon_z + \epsilon_n - \mu)} + 1)} \quad (3)$$

with  $\mu$  the chemical potential ( $\approx \epsilon_F$  at low  $T$ ). In terms of the one-dimensional Fermi integrals  $F_l(\gamma) =$

$\int_0^\infty dz z^l 1/(e^{z-\gamma} + 1)$ , the cooling power for the waveguide case is

$$C_V \frac{dT}{dt} = -\frac{2A}{d^2 h \beta^2} p_W \quad (4)$$

where

$$p_W \equiv \sum_{n=1}^{\infty} [F_1(\alpha_n) - (\xi - \beta \epsilon_n) \ln(1 + e^{\alpha_n})]. \quad (5)$$

The filling of the bands in the channels is determined by  $\alpha_n = \beta(\mu - \epsilon_n)$  with  $\beta = (k_B T)^{-1}$  and  $\xi = \beta(\partial E/\partial N)_{T,V}$  is a property of the gas in the cooling chamber (B1). For cooling  $p_W$  must be positive. If  $\gamma = \beta\mu$  and  $t = T/T_F$  then the relation between density and chemical potential leads to the usual transcendental equation for  $\mu$ :

$$\frac{3}{2} t^{3/2} F_{1/2}(\gamma) = 1 \quad (6)$$

Further, taking the derivative of  $E$  versus  $N$  gives

$$\xi = \frac{G_{3/2}(\gamma)}{G_{1/2}(\gamma)} \quad (7)$$

where  $G_l(\gamma) = \int_0^\infty dz z^l e^{z-\gamma}/(e^{z-\gamma} + 1)^2 = l F_{l-1}$ . Given a value of  $t$  from the temperature and the concentration, we must solve Eq. (6) for  $\gamma$ ; we put that into Eq. (7) to get  $\xi$ . We find it convenient to introduce the ratio of lowest band edge to Fermi energy  $y \equiv \epsilon_0/\epsilon_F$  so that  $\beta \epsilon_n = y \eta_n/t$ .

For numerical calculations we fix  $T$  at 15 mK as a reasonable incoming temperature in the cooling cycle. Then, if the number of holes in the membrane is  $M = 10^{11}$ , the prefactor in Eq. (4) is 13  $\mu\text{W}$ . In the case of wide holes (where waveguide flow becomes effusion), the sum over bands in Eq. (4) becomes a double integral over transverse momenta and in this continuum limit we find

$$p_W \approx \frac{\pi}{2} \frac{t}{y} [F_2(\gamma) - \xi F_1(\gamma)] \quad \text{continuum limit} \quad (8)$$

We have made no assumptions here about whether the system is degenerate or not—just that the channel states are now continuous. If the factor in square brackets is positive, then it says that the smaller  $y$ , the larger is the cooling. This result stems from just having bigger and bigger holes, allowing more hot atoms out. But of course there is a limit to how big the holes can be to maintain a pressure differential across the membrane.

Next consider the highly degenerate limit of the last form, for which  $F_n = \gamma^n/n$  and  $\xi = \gamma$ , so that

$$p_W \approx -\frac{\pi}{12} \frac{t}{y} \gamma^3 \quad \text{degenerate continuum limit} \quad (9)$$

which agrees with a direct calculation—no cooling occurs in that case, because there are many channel states below the Fermi energy allowing low energy atoms to escape.

Indeed the gas left behind can then end with a higher temperature.

For the fully classical limit we take  $e^\gamma$  to be very large so that

$$p_W \approx \frac{\sqrt{\pi}}{3} \frac{1}{t^{1/2}y} \text{ classical continuum limit.} \quad (10)$$

This result again agrees with the textbook classical effusion result.

In Fig. 2 we show numerical results based on Eq. (5). It is useful to plot  $yp_W$  rather than  $p_W$  because then all the plots collapse onto a single curve for  $t > 1$ , as implied by Eq. (8). The  $t$  dependence for large  $t$  is  $t^{-1/2}$  and we are indeed in the asymptotic region described by Eq. (10). We see that  $y = 1$  is a special point with the lowest band edge lined up with the Fermi energy. (Recall however, that the cooling power prefactor, which is not included in  $p_W$  contains a  $T^2$  so all cooling powers go to zero with decreasing  $T$ .) When the lowest band edge is lower than the Fermi energy ( $y < 1$ ) we lose cooling at very low  $T$  (as in Eq. (9)), but surprisingly we *regain* it for larger  $T$  due to the contributions of the higher bands. Finally for all band edges above the Fermi energy ( $y > 1$ ) we get cooling for all  $T$ , but, because the curves in the figure are multiplied by the factor  $y$ , the actual cooling power is diminished when we divide out this factor, and these cases are not as useful.

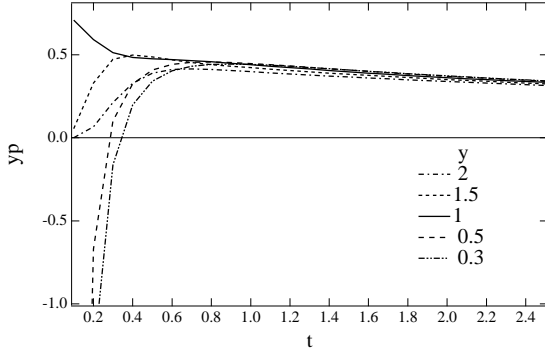


FIG. 2: Reduced waveguide cooling power as a function of temperature parameter  $t$  for various band-edge  $y$  values for a square cross section. Because of the multiplicative factor  $y$  all curves collapse onto the same classical limit curve at large  $t$ .

The parameters we have used are not independent. Because  $T_F$  depends on  $x^{2/3}$ , where  $x$  is the concentration, and  $\epsilon_0$  depends on  $d^{-2}$ , then for  $T = 15$  mK we find the relation

$$t = 1.8 \times 10^{-2} y (d(\text{nm}))^2 \quad (11)$$

From the plot, we see that if we have  $y < 1$  we get a  $1/y$  enhancement in cooling power, but to gain that we must also have  $t \gtrsim 0.4$  to avoid heating. For  $y > 1$  we can

go to lower  $t$  values but we lose cooling power because of the  $1/y$  factor and because of the dip in the curves.

Consider  $d = 10$  nm,  $M = 10^{11}$  which is perhaps within practical reach. The value  $t = 0.4$  ( $T_F = 35$  mK and  $x = 0.2\%$ ) implies  $y = 0.2$  and the cooling would be on the order of a  $50 \mu\text{W}$  in this ballistic-flow waveguide case. Such a result would be quite a remarkable cooling rate, but the assumption of perfectly smooth walls is optimistic. However, the inevitable coating of the pore walls by a couple of layers of solid  $^4\text{He}$  will enhance the smoothness.

It perhaps seems more likely that the walls of the channel would cause scattering, including back scattering, interband transitions, etc. Thus we consider next a simple wall-limited diffusion or Knudsen flow model. Our starting point is a kinetic equation for the distribution function in collision-time approximation[11],[12]

$$\mathbf{v}_p \cdot \nabla n_p = -\frac{1}{\tau} \delta n_p \quad (12)$$

where  $\tau$  is the time between collisions with the wall. The left side can be written as  $-v_{pz} \partial n_p^{(0)} / \partial \epsilon_p \partial \mu / \partial z$ , where  $n_p^{(0)}$  is the local equilibrium distribution function and  $\delta n_p$  is the correction to local equilibrium. The gradient in chemical potential  $\partial \mu / \partial z$  is proportional to the gradient in density. A much more rigorous approach to such a kinetic equation is described in Ref. [13]. We consider again the case of a very narrow channel containing banding. We solve for  $\delta n_p$  and use it to compute, say, the particle flux as

$$J_N = \frac{2}{d^2 h} \sum_n \int_{-\infty}^{+\infty} dp_z \frac{p_z}{m} \delta n_p \quad (13)$$

From Eq. (1) the cooling power is found to be

$$C_V \frac{dT}{dt} = - \left( \frac{2A}{d^2 h \beta^2} \right) p_K \quad (14)$$

where we have introduced the same prefactor as in the waveguide case so that

$$p_K = d \frac{d\gamma}{dz} \left[ \frac{t}{y \langle \eta_n \rangle} \right]^{1/2} \sum_n [G_{3/2}(\alpha_n) - \zeta_n G_{1/2}(\alpha_n)] \quad (15)$$

with  $\zeta_n = \xi - y \eta_n / t$ . In this equation the collision time  $\tau$  is written in terms of the *transverse* velocities in a channel. That is, we write  $\tau \sim d / (2\bar{v})$  with  $\bar{v} = \sqrt{2 \langle \epsilon_n \rangle} / m$  where

$$\langle \epsilon_n \rangle \equiv \sum_n \epsilon_n \int_{-\infty}^{+\infty} dp_z n_p^{(0)}(\epsilon_n) \left[ \sum_n \int_{-\infty}^{+\infty} dp_z n_p^{(0)} \right]^{-1} \quad (16)$$

If we divide this quantity by the lowest band edge, then we have  $\langle \epsilon_n \rangle / \epsilon_0 = \langle \eta_n \rangle$  used in Eq. (15).

We have to evaluate the derivative  $d\gamma/dz$ . What we mean by this quantity is  $d\gamma/dz = (d\gamma/dn)(dn/dz)$  since

it is the gradient in density  $n$  that drives the flow. We can find this by taking the derivative of the self-consistent expression, Eq. (6). In that equation  $t$  depends on  $n$  because  $\epsilon_F$  does. We find

$$\frac{d\gamma}{dz} = \frac{2t}{3n} \frac{F_{1/2}}{tG_{1/2}} \frac{dn}{dz} \approx \frac{4}{9} \frac{G_{3/2}}{G_{1/2}} \frac{1}{L} = \frac{4}{9} \xi \frac{1}{L} \quad (17)$$

where in the middle form we have taken the density gradient as  $n/L$  with  $L$  the length of a channel. What we compute then is

$$p_K = \frac{4}{9} \frac{d}{L} \xi \left[ \frac{t}{y \langle \eta_n \rangle} \right]^{1/2} \sum_n [G_{3/2}(\alpha_n) - \varsigma_n G_{1/2}(\alpha_n)]. \quad (18)$$

We again plot the product,  $y p_K$ . The results are shown in Fig. 3; note that the ratio  $d/L$  is *not* included in the plots. The curves differ from those of the waveguide in a several ways. The curves are always positive, i.e., represent cooling, and the  $y = 1$  degenerate case diverges (the  $T^2$  in the prefactor cures that). We have checked the numerical calculations by doing various limiting relations analytically. For example, the continuum (small  $y$ ) case in the limit  $t \rightarrow 0$  is  $p_K \rightarrow 2.41d/Ly$ .

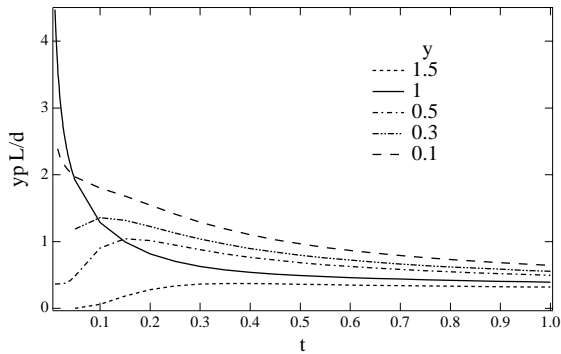


FIG. 3: Reduced Knudsen cooling power as a function of temperature parameter  $t$  for various  $y$  values for a square cross section. The factor of  $d/L$  in the overall cooling power is not included here.

As in the waveguide case, small  $y$  gives amplified cooling. Indeed, for  $M = 10^{11}$  holes,  $y = 0.1$  gives  $p_K L/d \approx 1.6/y = 16$ . Again consider  $d = 10$  nm. The cooling power is  $13 \mu\text{W} \times 16d/L = 208d/L \mu\text{W}$ . A membrane width of  $L = 1 \mu\text{m}$  gives a cooling of just  $2.1 \mu\text{W}$ . By Eq. (11) we have  $t = 0.18$  with the  $y$  value chosen or  $T_F = 83$  mK and  $x = 0.5\%$ . This cooling power might be useful if the width  $L$  used is not too optimistic. The cooling would be enhanced if a larger  $M$  value were available. For small  $y$  values the continuous degenerate limit gives an upper limit on the cooling power within these conditions. We have

$$p_K \lesssim 13 \frac{2.4}{y} \frac{d}{L} \mu\text{W} = \frac{31.2}{y} \frac{d}{L} \mu\text{W} \quad (19)$$

In summary, we have examined here the physics of particle flow through narrow pores and estimated the possibility of cooling by this method and have found some potential for success. As nano-technology improves, the possibilities may increase. A very smooth-walled channel that would provide waveguide type flow would give the greatest cooling. The more probable situation of Knudsen flow, while providing cooling over all parameter ranges, has the factor  $d/L$  reducing the cooling power. However, even that circumstance does not make it impossible. The numerical results give hope that this approach can lead to a add-on device to extend the range of a dilution refrigerator. Experiments are being planned to test the potential of the method.[14] While we have considered the possibility of a practical application of nanopores here, experiments on this kind also provide interesting physics, namely detecting quantum size effects in the narrow channels and the resultant restriction of states as already seen experimentally.[2, 3]

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